ISRAEL JOURNAL OF MATHEMATICS, Vol. 69, No. 2, 1990

THE THEOREMS OF BETH AND CRAIG IN ABSTRACT MODEL THEORY, III: Δ-LOGICS AND INFINITARY LOGICS

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ABSTRACT

We give a general technique on how to produce counterexamples to Beth's definability (and weak definability) theorem. The method is then applied for various infinitary, cardinality quantifier logics and Δ-closure of such logics.

§1. Introduction

For unexplained terms, background and history see the excellent representation in Makowsky [Ma]. We show that in many cases, the Beth theorem and weak definability fail. This may seem less "obviously true, just a proof is needed" now than in 1975, when this was done and should have been written. The reason is that then it was not unreasonable to think the subject would continue to produce counterexamples only. But, by Mekler–Shelah [MkSh, 166], it is consistent that the logic $L(\exists^{\geq \aleph_1})$ has W. (= weak) Beth. By [Sh199] INT[$L(Q^{\mathrm{cf}}_{\leq \aleph_j})$, $L(aa)$] holds and the Beth closure of $L(Q^{\mathrm{cf}}_{\leq 2^{\aleph_0}})$ (cofinality at most continuum) is compact.

In §2 we concentrate on counterexamples to W. Beth, and in §3 on counterexampes to Beth for Δ-closure of $L_{\infty,\omega}$.

In §2 we give sufficient conditions for the failure of Beth and Weak Beth (in 2.1–2.4B). This involves $\mathrm{Pr}^{x}_{\mathcal{L}_1,\mathcal{L}_2}(\mu, \theta, \sigma)$ (see Definition 2.1). So we deal

† The author would like to thank the United States–Israel Binational Science Foundation for partially supporting this research, and Alice Leonhardt for the beautiful typing.

This work (except 3.10 and variations) was carried out in 1975.

Publication No. 113.

Received May 26, 1988 and in revised form July 26, 1989

with finding instances of those properties for logics of the form $L_{\Theta,\kappa}$ (in 2.5) and then discuss why we cannot have some desirable cases (see 2.6–2.9); we state our conclusion on the failure of Beth and Weak Beth for logics of the form $L_{\Theta,\kappa}$ in 2.10. In 2.11–2.15 we look at properties of logics related to the properties concerning existence of models with automorphisms.

In §3 we show Δ-closures do not satisfy Beth. In 3.1–3.2 we give a sufficient (quite general) condition for the failure of Beth (or Weak Beth) for the Δ-closure of $\mathscr{L}$, which include using a counterexample to interpolation $(\sigma_1, \neg\sigma_2)$.

In 3.3 we get a specialized conclusion: the Beth theorem fails for $(\mathscr{L}, \Delta(L_{\infty,\omega}))$ if in $\mathscr{L}$ we can, essentially, have a pseudo elementary class separating two regular cardinals as cofinality. For this we rely on the abstract theorem 3.2, but the main work is verifying the condition from there which is done in 3.9. Before this 3.4 gives a specific conclusion ($L_{\omega_2,\omega}$ and $L(\exists^{\geq\lambda})(\lambda = \mathrm{cf}\,\lambda > \aleph_0)$) which does not satisfy Beth even when we look for the explicit definition in $\Delta(L_{\infty,\omega})$. This is an example for the abstract conclusion: Δ-interpolation does not imply Beth.

In the end of 3.10, we use a forcing of Gitik to derive a universe where for regular $\mu < \lambda$, even in $\Delta(L_{\infty,\mu})$, we cannot find interpolants for $L(\exists^{\geq\lambda})$ (or any $\mathscr{L}$, $L(\exists^{\geq\lambda}) \leqq \Delta(\mathscr{L})$).

The main work as mentioned above is in 3.9, which is a kind of generalization of the Morley omitting type theorem, this time controlling cofinalities of "many" orders (many — a set of linear orders indexed by sets which have to be large themselves); for "few" orders see [Sh18].

I thank Janos Makowsky for his great involvement in this paper. The reader would do better to have a copy of [Ma]; for Theorem 2.5 — [Sh189]; for 2.7 — [Sh133], [Sh228]; for 2.8(1) — [Sh129]; for 2.9 — [Sh125]; for 3.9, stage G — [Sh-a], VII, §5; for 3.10 — Gitik [G].

A logic $\mathscr{L}$ is a function such that, for any vocabulary τ, $\mathscr{L}(\tau)$ is a set of sentences (so for M a τ-model) and $\psi \in \mathscr{L}(\tau)$, $M \models \psi$ or $M \models \neg\psi$; this is preserved by isomorphism. If not mentioned otherwise, $\mathscr{L}$ is closed under the obvious operations $\wedge$, $\neg$, $\exists x$ and substitution. Of course, $\tau_1 \subseteq \tau_2 \Rightarrow \mathscr{L}[\tau_1] \subseteq \mathscr{L}[\tau_2]$ and $\mathscr{L}$ commutes with renaming relation and function symbols.

1.1. Open Problems. (1) W. Beth $(L(\exists^{\geq\lambda}))$ (i.e. provably in ZFC)?

(2) For κ strong limit singular (or weakly compact) is there $\psi \in L_{\kappa^+,\omega}$ all of whose models have cardinality κ and are $L_{\infty,\kappa}$-equivalent (but there are at least two)? (see [Sh228]; try [Sh355] §7).

(3) Is $L(Q^{ec})$ (equal cofinality quantifier) compact?

(*Note*: 3.9 is an approximation if we restrict ourselves to a suitable class of cardinals).

Recall:

1.2. Definition. (1) $M + N$ or $[M, N]$ is the disjoint sum of M and N (e.g. consider them as structures of different sorts).

(2) For logics $\mathscr{L}_1$ and $\mathscr{L}_2$ let PPP$(\mathscr{L}_1, \mathscr{L}_2)$ (= pair preservation properties) mean that for models M and N, $\mathrm{Th}_{\mathscr{L}_1}(M+N)$ is determined by $\langle \mathrm{Th}_{\mathscr{L}_1}(M), \mathrm{Th}_{\mathscr{L}_2}(N)\rangle$.

(3) For logics $\mathscr{L}_1$ and $\mathscr{L}_2$, INT$(\mathscr{L}_1, \mathscr{L}_2)$ means:

if $\varphi_1 \in \mathscr{L}_1(\tau_1)$, $\varphi_3 \in \mathscr{L}_1(\tau_2)$ and $\vDash \varphi_1 \rightarrow \varphi_3$, *then* for some $\varphi_2 \in \mathscr{L}_2(\tau_1 \cap \tau_2)$, $\vDash \varphi_1 \rightarrow \varphi_2$ and $\vDash \varphi_2 \rightarrow \varphi_3$.

(4) $\varphi = \varphi(P, \bar{R}) \in \mathscr{L}(\{P\} \cup \bar{R})$ is a Beth definition (W. Beth definition) *if* for every $\bar{R}$-model M there is at most one [exactly one] $P^M \subseteq |M|$, $(M, P) \vDash \varphi$.

(5) For logics $\mathscr{L}_1$ and $\mathscr{L}_2$, Beth$(\mathscr{L}_1, \mathscr{L}_2)$ [or W. Beth $(\mathscr{L}_1, \mathscr{L}_2)$] means that:

if $\varphi(P, \bar{R}) \in \mathscr{L}_1(\{P\} \cup \bar{R})$ is a Beth definition (W. Beth definition) of P, then some $\psi(x, \bar{R}) \in \mathscr{L}_2(\bar{R})$ is an explicit definition of P, i.e.

$$M = (A, \bar{R}^M, P^M) \vDash \varphi \Rightarrow P^M = \{a : M \vDash \varphi[a, \bar{R}]\}.$$

(6) INT$(\mathscr{L})$ means INT$(\mathscr{L}, \mathscr{L})$ and Beth$(\mathscr{L})$ means Beth$(\mathscr{L}, \mathscr{L})$ and W. Beth$(\mathscr{L})$ means W. Beth$(\mathscr{L})$.

(7) $\mathscr{L}_1 \leqq \mathscr{L}_2$ if, for every vocabulary τ, $\mathscr{L}_1(\tau) \subseteq \mathscr{L}_2(\tau)$.

§2. Sufficient conditions for Beth failure and applications

2.1. Definition. (1) $\mathrm{Pr}^a_{\mathscr{L}_1, \mathscr{L}_2}(\mu, \theta, \sigma)$ [$\mathscr{L}_1$ a logic, σ and θ are infinite cardinals, μ a cardinal] means (compare with 1.2):

(*) there is a sentence $\psi \in \mathscr{L}_1$ such that:

- (i) ψ has only rigid models,
- (ii) ψ has exactly μ models up to isomorphism,
- (iii) every model of ψ has cardinality $\leqq \theta$ (remember $\theta \geqq \aleph_0$),
- (iv)^a there is $M \vDash \psi$, and $a, b \in M$, $a \neq b$ such that $(M, a) \equiv_{\mathscr{L}_2} (M, b)$,
- (v) the vocabulary of ψ has cardinality $< \sigma$.

(2) $\mathrm{Pr}^b_{\mathscr{L}_1, \mathscr{L}_2}(\mu, \theta, \sigma)$ means as in (1), replacing (iv)^a by:

(iv)^b there are models M_0 and M_1 of ψ not isomorphic but $\mathscr{L}_2$-equivalent (so $\mu > 1$).

(3) $\Pr^c_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ means as in (1) when we replace (iv)^a by:
(iv)^c if M_1 and M_2 are models of ψ then $M_1 \equiv_{\mathscr{L}_2} M_2$.
(4) $\Pr^d_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ means as in (1) but instead of (iv)^a:
(iv)^d we have: (iv)^c *and* (iv)^a for every $M \models \psi$.

2.2. Observation. (1) There is obvious monotonicity (in $\mathscr{L}_1$, $\mathscr{L}_2$, θ, σ); i.e. if $\mathscr{L}_1 \leqq \mathscr{L}_1'$, $\mathscr{L}_2 \geqq \mathscr{L}_2'$, $\theta \leqq \theta'$, $\sigma \leqq \sigma'$ and $x \in \{a,b,c,d\}$, then $\Pr^x_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ implies $\Pr^x_{\mathscr{L}_1',\mathscr{L}_2'}(\mu,\theta',\sigma')$.

(2) $\Pr^a_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ implies $\Pr^b_{\mathscr{L}_1,\mathscr{L}_2}(\mu',\theta,\sigma)$ for some μ', $\mu \leqq \mu' \leqq \mu+\theta$.

(3) If $\mu > 1$, $\Pr^c_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ implies $\Pr^b_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$.

(4) $\Pr^d_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ implies $\Pr^x_{\mathscr{L}_1,\mathscr{L}_2}(\mu,\theta,\sigma)$ for $x = a, c$.

Proof. (1)–(4). Use the same example.

2.3. Theorem. (1) *If* $\Pr^b_{\mathscr{L}_1,\mathscr{L}_3}(\mu,\theta,\sigma)$ *and* $\mathrm{PPP}(\mathscr{L}_2,\mathscr{L}_3)$ *and* $L_{\sigma,\omega} \leqq \mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$, then $\mathrm{Beth}(\mathscr{L}_1,\mathscr{L}_2)$ *fails*.

(2) *Suppose* $\Pr^a_{\mathscr{L}_1,\mathscr{L}_2}(1,\theta,\sigma)$, $L_{\sigma,\omega} \leqq \mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$ *and* $\mathrm{PPP}(\mathscr{L}_2,\mathscr{L}_3)$, *then* W. $\mathrm{Beth}(\mathscr{L}_1,\mathscr{L}_2)$ *fails*.

Proof. (1) Let M be a model of $\psi \in \mathscr{L}_1[\tau]$. For simplicity assume τ has no function symbols. We construct a $\tau \cup \{P\}$-structure $\mathfrak{B}$ in the following way: $\mathfrak{B} = M \times \{0\} \cup M \times \{1\}$, i.e. $\mathfrak{B}$ is the disjoint union of two copies of M.

For each n-ary relation symbol $R \in \tau$ let R^M be its interpretation in M. Now we put

$$R_i = \{((a_1,i),\ldots,(a_n,i)) : (a_1,\ldots,a_n) \in R^M\} \quad \text{and} \quad R^{\mathfrak{B}} = R_0 \cup R_1.$$

P is a unary predicte and $P^{\mathfrak{B}} = M \times \{0\}$. (Alternatively, use multi-sorted models.) Let F be a binary relation symbol not in $\tau \cup \{P\}$. Let φ be a $\tau \cup \{P,F\}$-sentence of $\mathscr{L}_1$ expressing that:

(1) the relativized structures on P and on $\neg P$ are models of ψ;
(2) F is a τ-isomorphism from P to $\neg P$;
(3) F is of order two, i.e. F^2 is the identity on P.

Clearly, φ defines F implicitly, since ψ has only rigid models, by (i) of 2.1. So assume, for contradiction, that $\mathrm{Beth}(\mathscr{L}_1,\mathscr{L}_2)$ holds. Let $\theta \in \mathscr{L}_2$ be an explicit definition of F. By 2.1(2), ψ has non-isomorphic models M and N that $M \equiv_{\mathscr{L}_3} N$. We use $\mathrm{PPP}(\mathscr{L}_2,\mathscr{L}_3)$ to conclude that

(*) $M + N \equiv_{\mathscr{L}_2} M + M$

where the first sort represents the interpretation of P and the second sort the interpretation of $\neg P$. Clearly, $M+M$ has an expansion satisfying φ

and therefore θ defines a τ-isomorphism between the two sorts. So $(*)$ says that θ also defines a τ-isomorphism in $M+N$. This is expressed by a sentence in $L_{\sigma,\omega}$ (as $\sigma > |\tau|$) hence in $\mathscr{L}$. Therefore $M \cong N$ contradicting the choice of M, N.

(2) Like part (1), only in the end do we use $(M, a) \equiv_{\mathscr{L}_2} (M, b)$, $a \neq b$ where M is (the) model of ψ.

2.3A. Remark. The closure properties of the (set of sentences of the logics) $\mathscr{L}_1$ are very weak: we use a conjunction of a sentence from $L_{\sigma,\omega}$ (the isomorphism) and two copies of ψ.

2.4. Theorem. *If* $\mathrm{Pr}^b_{\mathscr{L}_1,\mathscr{L}_3}(\mu, \theta, \sigma)$, $\kappa^+ \geqq \sigma + \mu^+ + \theta^+$, $L_{\kappa^+,\omega} = \mathscr{L}_0 \leqq \mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$ *and* $\mathrm{PPP}(\mathscr{L}_2, \mathscr{L}_3)$, then Weak Beth *for* $(\mathscr{L}_1, \mathscr{L}_2)$ *fails*.

2.4A. Remark. (1) We assume (implicitly) the vocabulary consists of predicates and function symbols with finite arity. If we want to delete this assumption we should demand in 2.1, for every model M of φ, that

$$\| M \| + \sum \{ \| R^M \| : R \text{ a predicate of } M\} + \sum_f |\{(\bar{a}, b) : F^M(\bar{a}) = b\}| : F^M \text{ a function of } M\}|$$

is $\leqq \theta$.

(2) We choose below a proof which does not require that $\mathscr{L}_i$ are closed under infinitary operations; just include $L_{\kappa^+,\omega}$ and closure under: relativization for $\mathscr{L}_1$ (definition of ψ_1) and finitary operations for $\mathscr{L}_2$ (see end of argument).

(3) I do not see many closure requirements on $\mathscr{L}$ (except isomorphism of vocabularies); only substitute in

$$\underset{\text{one sort}}{\exists x} \ \underset{\text{second sort}}{\exists y} \left[\varphi(x, y) \wedge \underset{\text{first sort}}{(\forall x')} (x \leqq x')\right] ?$$

Proof of 2.4. Let ψ exemplify $\mathrm{Pr}^b_{\mathscr{L}_1,\mathscr{L}_2}(\mu, \theta, \sigma)$.

Let M_i $(i < \mu)$ be a complete list up to isomorphism with no repetition of models of ψ, and each has vocabulary τ, so $|\tau| \leqq \kappa$ (as $|\tau| < \sigma$, $\sigma \leqq \kappa^+$) and w.l.o.g. $M_0 \equiv_{\mathscr{L}_3} M_1$. W.l.o.g. the universe of each M_i is $\{\beta : \alpha_i \leqq \beta < \alpha_{i+1}\}$ and $\langle \alpha_i : i \leqq \mu \rangle$ is increasing continuous, $\alpha_0 = 0$. Let $\alpha(*) = \alpha_\mu$, so $\alpha(*) < \kappa^+$. Let us define a model M^*:

(a) R^{M^*} (R a two-place predicate) is

$$\{\langle i, \alpha \rangle : \alpha_i \leqq \alpha < \alpha_{i+1} \text{ and } i < \mu\};$$

(b) *if* P is an $n(P)$-place predicate of τ, then Q_P is an $(n(P)+1)$-place predicate and

$$Q_P^M = \{\langle i, a_1, \ldots, a_{n(P)}\rangle : \langle a_1, \ldots, a_{n(P)}\rangle \in P^{M_i}\};$$

(c) similarly for every function symbol of F;

(d) $<^{M^*}$ is $\{\langle \alpha, \beta\rangle : \alpha < \beta < \alpha(*)\}$.

Let τ^* be the vocabulary of M^*. Let $\psi_0 \in L_{\kappa^+,\omega}(\tau^*)$ characterize M^* up to isomorphism. Let $[\tau_1, \tau_2]$ be the vocabulary of $[N_1, N_2]$ when N_l has vocabulary τ_l (e.g. each N_l finitely sorted). Let $\psi_1 \in \mathscr{L}_1([\tau^*, \tau])$ be such that:

$$[N_1, N_2] \models \psi_1 \Leftrightarrow [N_1 \models \psi_0 \text{ and } N_2 \models \psi].$$

Let F be a new unary function, and let $\psi_2, \psi_3 \in \mathscr{L}_{|\tau|^+,\omega}[[\tau^*, \tau] + \{F\}]$ say:

$[N_1, N_2] \models \psi_2$ *iff* F is a one-to-one function from N_2 into N_1 and there is $x \in N_1$ such that:
Rang $F = \{y : N_1 \models R[x, y]\}$ and for every predicate P of ψ, and $a_1, \ldots, a_{n(P)} \in N_2$,
$N_2 \models P(a_1, \ldots, a_{(P)})$ *iff*
$N_1 \models Q_P[x, F(a_1), \ldots, F(a_{n(P)})]$
(similarly for function symbols).

$[N_1, N_2] \models \psi_3$ *iff* F is as above, but the x there is $<^{N_1}$-first.

Now every model of ψ_1 can be expanded uniquely to a model of ψ_3 (equivalently to a model of $\psi_1 \wedge \psi_3$):

However, $[M^*, M_0] \equiv_{\mathscr{L}_1} [M^*, M_1]$ as $M_0 \equiv_{\mathscr{L}_3} M_1$ (by (iv)[b] of Definition 2.1(2) and PPP($\mathscr{L}_1, \mathscr{L}_3$)). If some $\varphi(x, y) \in \mathscr{L}_2[\tau^*, \tau]$ defines F for models of ψ_1 (as would be the case if W. Beth($\mathscr{L}_1, \mathscr{L}_2$) holds), then using $\varphi(x, y)$ we easily distinguish between $[M^*, M_0]$ and $[M^*, M_1]$. Now expand by the function; it is defined.

So one is a model of ψ_3, the other not. So W. Beth($\mathscr{L}_1, \mathscr{L}_2$) fails as required.

2.4B. Conclusion. If

(a) $L_{\sigma,\omega} \leqq \mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$,

(b) $\psi \in \mathscr{L}_1[\tau]$ has a unique model M up to isomorphism, $|\tau| < \sigma$, M is rigid, $\| M \| + |\tau| < \sigma$ and for some $a \neq b$ from M, $(M, a) \equiv_{\mathscr{L}_3} (M, b)$,

(c) PPP($\mathscr{L}_2, \mathscr{L}_3$),

then W. Beth($\mathscr{L}_1, \mathscr{L}_2$) *fails*.

Proof. By 2.4.

2.5. THEOREM. (1) *If* $1 < \mu \leqq \theta^\kappa = \theta \geqq \aleph_0$, $\kappa = \mathrm{cf}\,\kappa > \aleph_0$ *then* $\mathrm{Pr}^c_{L_{\theta^+,\omega}, L_{\infty,\kappa}}(\mu, \theta, \theta^+)$ (*hence* $\mathrm{Pr}^b_{L_{\theta^+,\omega}, L_{\infty,\kappa}}(\mu, \theta, \theta^+)$).

(2) *If* $\theta \geqq \kappa = \mathrm{cf}\,\kappa > \aleph_0$ then *for some* μ, $\theta \leqq \mu \leqq \theta^\kappa$, $\mathrm{Pr}^b_{L_{\theta^+,\omega}, L_{\infty,\kappa}}(\mu, \theta, \theta^+)$.

(3) *Really* $\mathrm{Pr}^b_{L_{\theta^+,\omega}, L_{\kappa^\infty,\omega}}(\mu, \theta, \aleph_0)$ *hold for* $x = a, b, c, d$ *provided that* $[x = b, d \Rightarrow \mu > 1]$ (with little effort, but this gives little more than the previous parts).

PROOF. Essentially from [Sh189].

(1) By [Sh189], Fact 3.10, p. 46 there is a smooth (κ-system) $\mathfrak{A}$, $\|\mathfrak{A}\| = \theta^\kappa = \theta$, with $E(\mathfrak{A}) = \mu$, every $h_{i,j}$ is onto G_i ($\kappa = \mathrm{cf}\,\kappa > \aleph_0$ is understood).

Now look at the proof of [Sh189], Fact 3.11, p. 47; it proves all we want.

We define models $M_{\mathbf{a}}$ for $\mathbf{a} \in \mathrm{Gr}(\mathfrak{A})$; note $|\text{vocabulary}\,(M_{\mathbf{a}})| \leqq \mu$.

By [Sh189], 3.12, p. 47.

$$M_{\mathbf{a}} \cong M_{\mathbf{b}} \text{ iff } \mathbf{a} - \mathbf{b} \in \mathrm{Fact}(\mathfrak{A}).$$

By [Sh189], 3.13, p. 48 any two $M_{\mathbf{a}}$, $M_{\mathbf{b}}$ are $L_{\infty,\kappa}$-equivalent. Let ψ be the sentence expressing (*) of [Sh189], 49^5. By [Sh189], 44^7–44^{13} $\{M_{\mathbf{a}} : \mathbf{a} \in \mathrm{Gr}(\mathfrak{A})\}$ is the class of models of ψ up to isomorphism. As

$$E(\mathfrak{A}) = \mathrm{Gr}(\mathfrak{A})/\mathrm{Fact}(\mathfrak{A})$$

we finish.

(2) Like above using [Sh189], 3.8 instead 3.10.

(3) Not used, and is easy, so we have left it to the reader.

2.6. DEFINITION. For a model M

(1) for logic $\mathscr{L}$

$$\mathrm{no}_{\mathscr{L}}(M) = \{N/\cong : N \cong_{\mathscr{L}} M,\ \|N\| = \|M\|\},$$

(2) when $\mathscr{L} = L_{\infty,\|M\|}$ we omit it.

2.7. THEOREM. (1) If μ *is a weakly compact cardinal* $\lambda \leqq \mu$ *or* $\lambda = 2^\mu$ then $\mathrm{Pr}^a_{L_{\mu^+,\mu}, L_{\infty,\mu}}(\lambda, \mu, \aleph_0)$.

(2) *If* μ *is a singular strong limit cardinal*, $\lambda < \mu$ *or* $\lambda = 2^\mu$, *then* (1)*'s conclusion holds.*

PROOF. (1) By [Sh133] there is a model M, $\|M\| = \mu \geqq |\text{vocabulary}\,(M)|$ and $\mathrm{no}(M) = \lambda$ (see Definition 2.6) (decreasing the vocabulary is easy, see [Sh189]).

(2) Use [Sh228].

2.7A. REMARK. Actually, in 2.7 we get $\mathrm{Pr}^d_{L_{\mu^+,\mu}, L_{\infty,\mu}}(\lambda, \mu, \infty)$.

2.8. LEMMA. (1) *Suppose* $V = L$ *and* μ *is regular uncountable not weakly compact, then* $\mathrm{Pr}^b_{L_{\infty,\mu}, L_{\infty,\mu}}(\leqq \mu, \mu, \mu^+)$ *fail (hence by monotonicity* $\mathrm{Pr}^x_{L_{\kappa^+,\mu}, L_{\infty,\mu}}(\leqq \mu, \mu, \mu^+)$).

(2) *If* $\mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$ *and*

(*) *there is no model* M, $\| M \| + |\text{vocabulary}(M)| = \theta = \| M \|$, $1 < \mathrm{no}_{\mathscr{L}_2}(M) \leqq \mu$,

then $\mathrm{Pr}^x_{\mathscr{L}_1, \mathscr{L}_3}(\leqq \mu, \theta, < \infty)$ *fail for* $x = a, b, c, d$.

REMARK. The assumption $V = L$ is necessary in 2.8; see 2.9.

PROOF. (1) Follows from (2) as (*) of part (2) of this lemma holds (for $\lambda = \aleph_1$, by Palyutin; for $\lambda \geqq \aleph_1$, by [Sh129]).

(2) By 2.2 w.l.o.g. $x = b$, and let ψ exemplify $\mathrm{Pr}^b_{\mathscr{L}_1, \mathscr{L}_2}(\leqq \mu, \mu, < \infty)$ and we shall eventually get a contradiction.

Let M_0 and M_1 be as in (iv)^b of Definition 2.1. So in $K = \{M : \| M \| = \mu, M \equiv_{L_{\infty,\mu}} M_0\}$ there are at least two non-isomorphic models. By (*), in K there are at least μ^+ non-isomorphic models, but as $\psi \in \mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$, all of them are models of ψ, so (ii) of Definition 2.1 (applied to ψ) is contradicted.

2.9. FACT. In some generic extension of V:

(i) $V \vDash$ G.C.H.,

(ii) for some $\psi \in \mathscr{L}_{\omega_2, \omega}$, all models M of ψ are $\mathscr{L}_{\infty, \omega_1}$-equivalent, and $\mathrm{no}_{\mathscr{L}_{\infty,\omega_1}}(M) = \aleph_0$.

PROOF. By [Sh125]. (Let G be a strongly $\aleph_1$-free abelian group of cardinality $\aleph_1$. Let $H = G \times \mathbf{Z}$, h the natural projection from H onto G. Considering H, G, $\mathbf{Z}$ as abelian groups, let M be $[H, G, \mathbf{Z}]$ enriched by h and individual constant for every $c \in G \cup \mathbf{Z}$.) Now as G is strongly $\aleph_1$-free, if $h' : H' \to G$ is a homomorphism onto G, $\mathrm{Ker}\, h = \mathbf{Z}$ then $[H', G, \mathbf{Z}; h] \equiv_{L_{\infty,\omega_1}} M$. Easily (if you understand the definitions) $\mathrm{no}(M) = |\mathrm{Ext}(G, \mathbf{Z})|$. But by [Sh125], $|\mathrm{Ext}(G, \mathbf{Z})|$ can be $\aleph_0$.

2.10. CONCLUSION. (1) If κ is regular, $> \aleph_0$, then W. Beth$(L_{(2^\kappa)^+, \omega}, L_{\infty,\kappa})$ fails.

(2) If κ is weakly compact $> \aleph_0$ or κ strong limit, $\aleph_0 < \mathrm{cf}\,\kappa < \kappa$ then W. Beth$(L_{\kappa^+,\kappa}, L_{\infty,\kappa})$ fails.

(3) If $\theta \geqq \kappa = \mathrm{cf}\,\kappa > \aleph_0$ then Beth$(L_{\theta^+,\omega}, L_{\infty,\kappa})$ fails.

PROOF. (1) Use 2.4 for $\mathscr{L}_1 = L_{(2^\kappa)^+,\omega}$, $\mathscr{L}_2 = \mathscr{L}_3 = L_{\infty,\kappa}$; $\mu = 2^\kappa = \theta$, $\sigma = \mu^+$ (and κ there is 2^κ here); now we have to verify the hypothesis: $\mathrm{Pr}^b_{\mathscr{L}_1,\mathscr{L}_3}(\mu, \theta, \sigma)$ by 2.5(1); PPP($\mathscr{L}_2$, $\mathscr{L}_3$) is well known. (I think it is due to Malitz — see [Ma]).

(2) Like (1) but use 2.7 instead 2.5(1).

(3) Use 2.3(1) and 2.5(2).

REMARK. (1) So by 2.8 in 2.5 we cannot replace $\theta^\kappa = \theta$ by $\theta \geqq \kappa$. Can we replace it by $\theta = \kappa^{+\alpha}$? Note that the proof of 2.8 uses only the following consequences of $V = L$ on μ:

(a)$_\mu$ every stationary $S \subseteq \mu$ has a stationary subset S_1^* which does not reflect. Moreover, S_1^* satisfies a square principle:

$(*)_{S_1^*}$ there is $\langle C_\delta : \delta < \mu,\ \delta \text{ limit}\rangle$, C_δ is a club of δ disjoint to S_1^*, $\alpha \in C_\delta \ \&\ \alpha = \sup(C_\delta \cap \alpha) \Rightarrow C_\alpha = C_\delta \cap \alpha$;

(b)$_\mu$ for every stationary $S \subseteq \mu$ the weak diamond is satisfied (see Devlin–Shelah [DSh65], [Sh-b], Ch. XIV, §1).

Now those demands are not hard, e.g. define

(a)$'_\mu$ there is a closed unbounded subset C of θ and a function $h : C \to \theta$, $h(\alpha) < \alpha$ such that $\forall \alpha < \theta$

$$[h^{-1}(\{\alpha\}) \text{ does not reflect and it satisfies } (*)].$$

Now (a)$'_\mu \Rightarrow$ (a)$_\mu$, (a$'$)$_\mu$ holds in L, and if $V \models$ "(a)$'_\mu$" then (a)$'_\mu$ holds in any extension of V (in which μ is still a regular cardinal).

Now we continue [Sh199], §3.

2.11. DEFINITION. Let $\mathscr{L}_1 \subseteq \mathscr{L}_2$ be logics.

(i) We say that the pair of logics $\mathscr{L}_1$ and $\mathscr{L}_2$ has the Local Homogeneity Property, if for every τ-structure M and $c_1, c_2 \in M$ such that $\langle M, c_1\rangle \equiv_{\mathscr{L}_2} \langle M, c_2\rangle$ and every $\varphi \in \mathrm{Th}_{\mathscr{L}_1}(\langle M, c_1, c_2\rangle)$ there is a model $\langle N, c_1^N, c_2^N\rangle \models \varphi$ and a τ-automorphism g of N such that $g(c_1^N) = c_2^N$. If $\mathscr{L}_1 = \mathscr{L}_2$ we just say that $\mathscr{L}_1$ has the Local Homogeneity Property.

(ii) We say that $\mathscr{L}$ has the Local Automorphism Property, if for every τ-structure M and infinite subset $P \subseteq M$, every sentence φ of the theory $\mathrm{Th}_{\mathscr{L}}(\langle M, P\rangle)$ has a model $\langle N, P'\rangle$ which has an automorphism g of N such that $g \restriction P' \neq \mathrm{Id}$.

We now define a Local Automorphism Property for pairs of logics.

2.12. DEFINITION. We say that the pair of logics $(\mathscr{L}_1, \mathscr{L}_2)$ has the Local Automorphism Property, if for every τ-structure M, every infinite subset

$P \subseteq M$, and for every $a, b \in P$, $a \neq b$ such that $\langle M, a \rangle \equiv_{\mathscr{L}_2} \langle M, b \rangle$, every sentence φ of the theory $\mathrm{Th}_{\mathscr{L}_1}(\langle M, P \rangle)$ has a model $\langle N, P' \rangle$ which has an automorphism g of N such that $g \restriction P' \neq \mathrm{Id}$.

Note that we do not require in 2.11(i) that the automorphism interchanges a and b.

2.13. THEOREM. *Let $\mathscr{L}_i$, $i = 1, 2, 3$ be three logics such that $\mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$ and that* $\mathrm{INT}(\mathscr{L}_1, \mathscr{L}_2)$ *and* $\mathrm{PPP}(\mathscr{L}_2, \mathscr{L}_3)$ *hold. Then the pair of logics $(\mathscr{L}_1, \mathscr{L}_3)$ has the Local Homogeneity Property.*

PROOF. Let M and $c_1, c_2 \in M$ be as in the hypothesis of the Local Homogeneity Property (see Definition 2.11(1)). Let M', c_1', c_2' be a disjoint copy. Put $N = [M, M']$. Put

$$T = \mathrm{Th}_{\mathscr{L}_2}(\langle N, c_1, c_2, c_1' \rangle) = \mathrm{Th}_{\mathscr{L}_2}(\langle N, c_1, c_2, c_2' \rangle).$$

The equality holds because of $\mathrm{PPP}(\mathscr{L}_2, \mathscr{L}_3)$. Let c_1, c_2 be constant symbols with interpretations c_1, c_2 and c be a constant symbol with interpretation c_1' or c_2' respectively. Let F_1, F_2 be two new function symbols. Let ψ_i $(i = 1, 2)$ be the sentence which says that F_i is a τ-isomorphism (modulo name changing) mapping the first sort into the second sort which maps c_i into c. Clearly, $T \cup \{\psi_i\}$ has a model for each $i = 1, 2$. Let $\varphi \in T \cap \mathscr{L}_1[\tau]$. If $\{\varphi, \psi_1, \psi_2\}$ has a model $[M_1, M_1', F_1, F_2]$ we get the required automorphism in M_1 from the composition of F_1 and F_2^{-1}. So asume that $\{\varphi, \psi_1, \psi_2\}$ has no model. We now apply $\mathrm{INT}(\mathscr{L}_1, \mathscr{L}_2)$ and find $\theta \in \mathscr{L}_2[\tau]$ such that $\psi_1 \rightarrow \varphi \wedge \theta$ and $\varphi \wedge \theta \rightarrow \neg \psi_2$ are both valid. But $\varphi \wedge \theta \in T$ and T has models which allow expansions satisfying ψ_1 and also models which allow expansions satisfying ψ_2, a contradiction.

2.14. THEOREM. *Suppose*

(a) $\mathscr{L}_1 \leqq \mathscr{L}_2 \leqq \mathscr{L}_3$.

(b) $\mathrm{PPP}(\mathscr{L}_2, \mathscr{L}_3)$.

(c) *If $\psi \in \mathscr{L}_2[\tau \cup \{P, Q\}]$ (τ-vocabulary, P, Q predicates) and, for every τ-model M,*

$$|\{Q : \text{for some } P\ (M, Q, P) \vDash \psi\}| \leqq 1$$

then *for some $\varphi(x) \in \mathscr{L}_1(\tau)$ for every τ-model M*

$$(M, Q, P) \vDash \varphi \Rightarrow Q = \{a : M \vDash \varphi[a]\}.$$

Then $(\mathscr{L}_1, \mathscr{L}_3)$ *has the weak Local Automorphism Property (which means, in* 2.12(1) *we add to the hypothesis $|P^M| > h(|\tau|)$, h a function depending on $\mathscr{L}$).*

Of course,

2.15. Observation. If the pair $(\mathscr{L}_1, \mathscr{L}_2)$ has the Local Homogeneity Property then this pair has the Local Automorphism Property.

§3. Δ-closure does not help Beth

Our main Theorem is 3.3, but most of the work is done in 3.9, and the interest is exposed in the conclusions 3.4 (specific logics) and 3.5 (counterexamples in abstract model theory).

Essentially the theorem says that $\Delta(\mathscr{L}_{\infty,\omega})$ is far from having the Beth property, even for implicit definitions in quite weak logics (like $L(\exists^{\geqq \aleph_1})$, $L_{\omega_2,\omega}$).

The proof uses the idea of the Morley omitting type theorem (in Stage I of proof of 3.9), Hutchington's idea of using a model with large cofinality so that it has many $\mathscr{L}_{\kappa,\omega}$-elementary submodels with cofinality $\aleph_0$ and with cofinality $\aleph_1$, and the proof of not Beth($\mathscr{L}$) (see the exposition [Ma]).

However, the following theorem isolates a sufficient condition for the failure of the Beth (weak Beth) Property for Δ-logics.

Recall:

3.0. Definition. For a logic $\mathscr{L}$, let $o(\mathscr{L})$ be the minimal cardinal $\lambda \geqq \aleph_0$ such that for every vocabulary τ and $\psi \in \mathscr{L}[\tau]$ for some $\tau' \subseteq \tau$ of cardinality $<\lambda$: if M_1 and M_2 are τ models and $M_1 \upharpoonright \tau' = M_2 \upharpoonright \tau'$ then $M_1 \models \psi \Leftrightarrow M_2 \models \psi$.

3.1. Definition. Let $\mathscr{L}$ be a logic with dependence number $o(\mathscr{L}) \leqq \mu$. Let $\sigma(P) = \sigma_1(P) \vee \sigma_2(P)$ be a $\mathscr{L}[\tau \cup \{P\}]$-formula and $\sigma_1(P)$, $\sigma_2(P)$ are contradictory.

(1) M is a (μ, κ)-counterexample for σ (strictly speaking for $\langle \sigma_1(P), \sigma_2(P), P \rangle$) if M has vocabulary τ, $|\tau| < \mu$ and: *for every* expansion of M to a τ^*-structure M^*, and $T \subseteq \mathrm{Th}_{\mathscr{L}}(M^*)$, where $|T| < \kappa$ (note $\tau \subseteq \tau^*$), $P \notin \tau^*$, and card$(\tau^*) < \mu$, *there are* τ^*-structures N_1 and N_2 such that:

(i) $N_l \models T$,

(ii) $N_l \models (\exists P)[(N_l, P) \models \sigma_l(P)]$ for $l = 1, 2$.

(2) If $\kappa = \aleph_0$, $\mu = o(\mathscr{L})$ we say M is an $\mathscr{L}$-counterexample for σ.

3.1A. Remark. We can use for $\sigma_l(P)$, $\sigma^*(P) \wedge \varphi(P)$, $\sigma^*(P) \wedge \neg\, \varphi(P)$.

3.2. Fact. Let $\mathscr{L}$ be a logic with dependence number $o(\mathscr{L}) \leqq \mu$. Let $\sigma(P)$ be a $\mathscr{L}(\tau \cup \{P\})$-formula which is an implicit definition of P.

(1) Assume that M is an $\mathscr{L}$-counter example for σ. Then $\Delta(\mathscr{L})$ does not have the Beth Property.

(2) If M is a (μ, ∞)-counterexample for σ then $\Delta(\mathscr{L})$ fails the FWROB (see [Ma]).

PROOF. (1) Let $\sigma(P)$ be a $\mathscr{L}(\tau \cup \{P\})$-formula which is an implicit definition of P. Assume for contradiction that there is an explicit definition for P given by a formula $\theta(x) \in \Delta(\mathscr{L})$. Since $\Delta(\mathscr{L})$ has the same dependence number as $\mathscr{L}$, there is a vocabulary $\tau^{\#}$ with $\tau \subseteq \tau^{\#}$ and $\mathrm{card}(\tau^{\#}) < \mu$ and $\theta_1(x)$, $\theta_2(x) \in \mathscr{L}(\tau^{\#})$ which forms the Δ-definition of θ. This means that for every τ-model M and $a \in M$ there is a unique $l \in \{1, 2\}$ such that for some expansion $M^{\#}$ of M to a $\tau^{\#}$-model, $M^* \models \varphi_l[a]$. Let for simplicity $\tau^{\#} \setminus \tau = \{R_i : i < \alpha\}$, and let R_i' be a predicate with $(n(R_i) + 1)$-places. Now let M be a $\mathscr{L}$-counterexample to $\sigma(P)$, for every $a \in M$ let $l(a) \in \{1, 2\}$, $R_i^a \subseteq {}^{n(R_i)}|M|$ for $i < \alpha$ be such that:

$$(M, R_i^a)_{i<\alpha} \models \theta_{l(a)}[a].$$

Define $R_i' = \{\langle a \rangle {}^\wedge \bar{b} : \bar{b} \in R_i^a \text{ where } a \in M\}$, $M^* \stackrel{\text{def}}{=} (M, R_i')_{i<\alpha}$. Define $\theta_l'(x)$ accordingly (substitute $R_i(x, y_1, \ldots, y_n)$ instead of $R_i(y_1, \ldots, y_n)$. Note: for $a \in M^* : M^* \models \theta_l'[a]$ *iff* $M^* \models \neg\, \theta_{3-l}[a]$. Let

$$T = \{\forall x[\theta_1'(x) \equiv \neg\, \theta_2'(x)]\} \cup \{\varphi : M^* \models \varphi, \varphi \text{ is } \mathrm{Sub}^{P(x)}_{\theta_l'(x)} \sigma_l(P) \text{ for } l = 1, 2\}.$$

So there are N_1 and N_2 as required in Definition 3.1, hence there are, for $l = 1, 2$, $P^l \subseteq N_l$, $(N_l, P^l) \models \sigma_l(P)$. By the choice of $[\theta_1(x), \theta_2(x)]$ and of M^*, for $l = 1, 2$ we have:

$$P^l = \{a \in N_l : N_l \models \theta_1[a]\}.$$

So $N_l \models \mathrm{Sub}^{P(x)}_{\theta_l'(x)} \sigma_l(P)$, hence $M^* \models \mathrm{Sub}^{P(x)}_{\theta_l'(x)} \sigma_l(P)$. Let $P^* = \{a \in M^* : M \models \theta_1[a]\}$, then $M^* \models \sigma_l(P^*)$ for $l = 1, 2$; but $\sigma_1(P)$, $\sigma_2(P)$ are contradictory; contradiction.

(2) Same proof with $T = \mathrm{Th}_{\mathscr{L}}(M^*)$.

3.3. THEOREM. $\mathrm{Beth}(\mathscr{L}, \Delta(L_{\infty,\omega}))$ *and, for every* μ, $\mathrm{FWROB}(\mathscr{L}, \Delta(L_{\mu,\omega}))$ *fail if for some distinct regular cardinals* $\kappa_1 \neq \kappa_2$ *and* $\theta_l \in \mathscr{L}(\tau_l)$ *where* $\tau_1 \cap \tau_2 = \tau_0 = \{<\}$, *we have for* $l = 1, 2$:

(a) $K_l \stackrel{\text{def}}{=} \{M \restriction \tau_0 : M \models \theta_l\}$ *are disjoint*,

(b) *for arbitrarily large* μ, *if* $(|M|, <^M)$ *is a linear order of cardinality* μ *and cofinality* κ_l *then* $M \in K_l$.

PROOF. If follows from the main lemma 3.9 proved below and 3.2. So we first draw conclusions, and then proceed to prepare for 3.9.

3.4. CONCLUSION. (1) If $\lambda > \aleph_0$ is regular, μ any cardinal, *then* Beth$[L_{\omega,\omega}(\exists^{\geq\lambda}), \Delta(L_{\infty,\omega})]$ and FWROB$[L_{\omega,\omega}(\exists^{\geq\lambda}), \Delta(L_{\mu,\omega})]$ fail.

(2) In particular this holds for the logic with the quantifier: there are uncountably many.

(3) Beth$[L_{\omega_2,\omega}, \Delta(L_{\infty,\omega})]$ and FWROB$[L_{\omega_2,\omega}, \Delta(L_{\mu,\omega})]$ fail.

(4) For $\mu > \aleph_1$, Beth$[\Delta(L_{\infty,\omega})]$ fails and Beth$[\Delta(L_{\mu,\omega})]$ and FWROB$[\Delta(L_{\mu,\omega})]$ fail.

3.5. COROLLARY. *Let* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *be two logics. Then*

(i) Δ-Int$(\mathscr{L}_1, \mathscr{L}_2)$ (see definition below) *does not imply* Beth$(\mathscr{L}_1, \mathscr{L}_2)$.

(ii) Δ-Int$(\mathscr{L}_1, \mathscr{L}_2)$ *does not imply* WFROB$(\mathscr{L}_1, \mathscr{L}_2)$.

The subsequent definitions and lemmas lead to a proof of 3.3.

3.5A. DEFINITION. Δ-Int$(\mathscr{L}_1, \mathscr{L}_2)$ means: if $\psi_l \in \mathscr{L}_1[\tau_l]$ for $l = 1, 2$, $\tau = \tau_1 \cap \tau_2$ and the class of τ-models is the disjoint union of

$$K_1 = \{M \restriction \tau : M \text{ a } \tau_1\text{-model satisfying } \psi_1\} \qquad \text{and}$$

$$K_2 = \{M \restriction \tau : M \text{ a } \tau_2\text{-model satisfying } \psi_2\}$$

then K_1 is the class of τ-models of ψ^* for some $\psi^* \in \mathscr{L}_2[\tau]$.

EXPLANATION. What is the point of the following game? We, on the one hand, want to build a type of indiscernibles to be able to control the cofinality. On the other hand, if the type, say, is bounded, we are lost. If the cofinality is weakly compact, we can use partition theorems, but maybe the class of weakly compact is bounded. But restricting ourselves to rapid sequences, solve the dilemma — it ensures unboundedness, and gives one type.

3.6. DEFINITION. Suppose M is a model, A a subset of $|M|$, $<$ a two-place relation which linearly orders A with no last element, and Φ a set of formulas $\varphi(\bar{x}, \bar{d})$ in the vocabulary of M with parameters from $|M|$.

(1) We say $\langle a_i : i < \alpha \rangle$ is *rapid* for $(A, <)$ over Φ inside M if, in the following game, player I has no winning strategy:

a play lasts α moves:

in the β-th move player I chooses $b_i \in A$ and player II chooses $c_i \in A$, such that $b_i < c_i$;

in the end player II wins if for every $\varphi(x_1, x_2, \ldots, x_n, \bar{d}) \in \Phi$, and $i_1 < i_2 < \cdots < i_n < \alpha$, we have:

$$M \models \varphi[a_{i_1}, \ldots, a_{i_n}, \bar{d}] \equiv \varphi[c_{i_1}, \ldots, c_{i_n}, \bar{d}].$$

(2) If $B \subseteq M$, Θ a set of formulas with vocabulary included in the vocabulary of M, we write (B, Θ) instead:

$$\Phi_{(B,\Theta)} \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{d}) : \bar{d} \subseteq B \quad \text{and} \quad \varphi(\bar{x}, \bar{y}) \in \Theta\}.$$

(3) We write (Ψ, n) instead of $\Phi_{(\Psi, n)}$ where

$$\Phi_{(\Psi, n)} \stackrel{\text{def}}{=} \{\varphi(x_1, \ldots, x_n, \bar{d}) : \varphi(x_1, \ldots, x_n, \bar{d}) \in \Psi\}.$$

(4) We write (B, Θ, n) instead of $\Phi_{(\Phi_{(B,\Theta)}, n)}$.

(5) If Θ is the set of first-order formulas in the vocabulary of M, we omit it.

(6) Writing "$\leqq n$" instead of "n" has the obvious meaning.

(7) Instead of "A", "$<$" and "B" we may write a formula which defines them. If the identity of "$<$" is obvious — we omit it.

3.7. Claim. (1) For any model $M, A, B \subseteq M$, $<$ (two-place relation) such that $(A, <)$ is linear with no last element, Φ are as above and $a_i \in A$ for $i < \alpha$, we have:

$\langle a_i : i < \alpha \rangle$ is rapid for $(A, <)$ over $(\Phi, \leqq, 0)$ inside M.

(2) If $\langle a_i : i < \alpha \rangle$ is rapid for $(A, <)$ over Φ^* inside M, $\Phi \subseteq \Phi^*$, $A \subseteq M$, $\langle i(\zeta) : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $< \alpha$, *then* $\langle a_{i(\zeta)} : \zeta < \xi \rangle$ is rapid for $(A, <)$ over Φ inside M.

(3) If $\langle a_i : i < \alpha \rangle$ is rapid for $(A, <)$ over (Φ^*, n) inside M, $n = 2$, $[x_1 < x_0] \in \Phi$, *then* $\langle a_i : i < \alpha \rangle$ is strictly increasing and $|\alpha| < \mathrm{cf}(A, <)$.

Proof. Easy.

3.8. Claim. If $\langle a_i : i < \alpha \rangle$ is rapid for $(A, <)$ over Θ inside M, $\Theta \subset \Psi$, $\mathrm{cf}(A, <)$ is bigger than $2^{|\Psi| + |\alpha|}$, *then* there are b_i $(i < \alpha)$ such that:

(a) $\langle b_i : i < \alpha \rangle$ is rapid for $(A, <)$ over Ψ inside M,

(b) for $\varphi(x_1, \ldots, x_n, \bar{d}) \in \Theta$, $i_1 < \cdots < i_n < \alpha$ we have
$M \models \varphi[b_{i_1}, \ldots, b_{i_n}, \bar{d}] \equiv \varphi(a_{i_1}, \ldots, a_{i_n}, \bar{d}]$.

Proof. The point is that any family of less than $\mathrm{cf}(A, <)$ strategies for player I can be combined to one strategy.

3.8A. Remark. In 3.9 (and 3.4), if we assume there is a class of measurables, we can simplify the proof and get $N \restriction \tau \prec M$. We can waive the notion of "rapid"; just let D_η be a normal ultrafilter on λ_η and $H(\eta)$ is such that $\{i : \eta^\wedge\langle i\rangle$ appears in $H(\eta)\} \in D_\eta$.

3.9. Main Lemma. *Suppose*

(a) $\psi \in L_{\mu^+,\omega}[\tau]$, $|\tau| \leqq \mu$ *(where* τ *is a vocabulary)*;

(b) $M^* \models \psi$, $(\tau =$ *vocabulary of* $\mu)$, $\psi \vdash \mathrm{Th}_{L_{\omega,\omega}}(M^*)$;

(c) T *a non-empty set of finite sequences of ordinals, closed under initial segments,* $\langle\ \rangle \in T$, *and for every* $\eta \in T$ *for some cardinal* $\lambda_\eta > \mu$:

$$\eta^\wedge\langle i\rangle \in T \Leftrightarrow i < \lambda_\eta;$$

(d) $T = P^{M^*}$, $<^M = \{(\eta^\wedge\langle i\rangle, \eta^\wedge\langle j\rangle) : \eta \in T, i < j < \lambda_\eta\}$, $F_l^M(\eta) = \eta \restriction l$, $P_n^{M^*} = \{\eta \in T : \lg(\eta) = n\}$, R *a coding function on* $\{\eta^\wedge\langle i\rangle : i < \lambda_\eta\}$;

(e) $\eta \neq \nu \Rightarrow \lambda_\eta \neq \lambda_\nu$ *for* $\eta, \nu \in T$; $\lambda_{\eta\restriction k} < \lambda_\eta$ *for* $k < \lg(\eta)$, $\eta \in T$ *and for simplicity* $[\eta, \nu \in T, \lg(\eta) < \lg(\nu) \Rightarrow \lambda_\eta < \lambda_\nu]$;

(f) *for* $\eta \in T$, λ_η *is regular and* $\geqq \beth_{(2^\mu)^+}(\mu_\eta)$ *where* $\mu_\eta \overset{\text{def}}{=} \mu + \Sigma\{\lambda_\nu : \lambda_\nu < \lambda_\eta\}$);

(g) $\kappa_1 \neq \kappa_2$ *are regular cardinals* $< \mathrm{Min}_{\eta \in T}\lambda_\eta$;

(h) $\tau_0 \subseteq \tau$, *the relation* $<$ *belongs to* τ_0, K_0 *a class of* τ_0*-models (closed under isomorphism), such that for every* $\eta \in T$: $(M^* \restriction \tau_0) \restriction \{\eta^\wedge\langle i\rangle : i < \lambda_\eta\}$ *belongs to* K_0; *and there is a* $\varphi(x) \in L^{[\tau_0]}_{\mu^+,\mu}$ *such that* $M^* \models \varphi[\eta]$ *say this;*

(i) *if* $M \in K_0$ *then* $<^M$ *linearly ordered* $|M|$;

(j) *if* $M \in K_0$, $l \in \{1, 2\}$, $\| M \| = \mu$ *and* $\mathrm{cf}(|M|, <^M) = \kappa_l$, then *for some* Q, $(M, Q) \in K_l$;

(k) K_l *is a class of* $(\tau_0 \cup \{Q\})$*-models closed under isomorphism and* $K_l' = \{N \restriction \tau_0 : N \in K_l\}$ *and* K_0', K_1' *are disjoint*;

then *there is a* τ*-model* N *of* ψ *and* $Q \subseteq \bigcup_n P_n^N$ *such that*:

(1) $P_0^N \subseteq Q$ *(necessarily* $|P_0^N| = 1$);

(2) *for every* $x \in P_n^N$ *letting*

$$N_x = (N \restriction \tau_0) \restriction \{y \in P_{n+1}^N : F_n(y) = x\},$$

the following are equivalent:

(i) $\mathrm{cf}(N_x, <) = \kappa_1$,

(ii) $\mathrm{cf}(N_x, <) \neq \kappa_2$,

(iii) $(N_x, Q \cap N_x) \in K_1$,

(iv) $(N_x, Q \cap N_x) \notin K_2$ *(hence* $N_x \notin K_1$).

Proof.

Stage A — Definition. (1) We say (M, H) is an α-approximation if:

(α) M is an expansion of M^* with Skolem functions (for first order formulas), having a name for every subformula of ψ;

(β) M has a vocabulary of cardinality $\leqq \mu$;

(γ) H has domain $\subseteq P^M$ $(=T)$ and, for every $\eta \in \mathrm{Dom}(H)$, $H(\eta)$ is a $<$-increasing sequence of members of $B^*_\eta = \{\eta\,\hat{}\,\langle i\rangle : i < \lambda_\eta\}$ of length $\beth_\alpha(\mu_\eta)^+ : \langle H_\eta(i) : i < \beth_\alpha(\mu_\eta)^+\rangle$;

(δ) $H(\eta)$ is rapid for $(B^*_\eta, <)$ over $A^*_\eta \stackrel{\text{def}}{=} \bigcup\{B^*_\nu : \lambda_\nu < \lambda_\eta\}$ inside M^*.

(2) If we omit α this means for some α. We call (M, H) full if $\mathrm{Dom}\, H = P^{M^*}$.

Stage B — Definition. (1) We say for approximations (M^l, H^l) $(l = 1, 2)$ that $(M^1, H^1) \leqq (M^2, H^2)$ if:

(α) M^2 is an expansion of M^1;

(β) $\mathrm{Dom}\, H^1 \subseteq \mathrm{Dom}\, H^2$;

(γ) for every $\eta \in \mathrm{Dom}\, H^1$, there is an increasing sequence $\langle i(\zeta) : \zeta < \lg(H^2(\eta))\rangle$ of ordinals $< \lg(H^1(\eta))$ such that the sequences $H^2(\eta)$, $\langle H^1_\eta(i(\zeta)) : \zeta < \lg(H^1(\eta))\rangle$ realize the same type in M^1 over A^*_η.

(2) We say $(M^1, H^1) \leqq_{\mathrm{pr}} (M^2, H^2)$ if in addition $\mathrm{Dom}\, H^1 = \mathrm{Dom}\, H^2$, $\lg(H^1(\eta)) = \lg(H^2(\eta))$.

(Both relations defined in this stage are partial order.)

Stage C — Definition. (1) We say an approximation (M, H) is (n, m, k)-indiscernible if:

for any distinct $\eta_1, \dots, \eta_m \in P^{M^*}_n$, and for $l = 1, \dots, m$, $\bar{b}_l$, $\bar{c}_l$ are increasing subsequences of $H(\eta_l)$ of length k then the types $\bar{b}_1\,\hat{}\cdots\hat{}\,\bar{b}_n$, $\bar{c}_1\,\hat{}\cdots\hat{}\,\bar{c}_m$ realizes over $\bigcup_{l \leqq n} P^{M^*}_l$ inside M^* are equal.

(2) If $k = m$ omit it.

Stage D — Fact. For every $\alpha < \beth_{(2^\mu)^+}(\mu)$ there is a full α-approximation (M, H).

Proof. Expand M^* by names for subformulas of ψ and then add Skolem functions. Lastly use 3.8 to define H, $\mathrm{Dom}\, H = T$.

Stage E — Fact. For any approximation (M^1, H^1), we can find M^2, H^2 such that:

(α) (M^2, H^2) is an approximation, $(M^1, H^1) \leqq_{\mathrm{pr}} (M^2, H^2)$;

(β) there is a predicate $R \in \tau(M^2) \setminus \tau(M^*)$, $R = \{(\eta, a) : a \in H^1(\eta)\}$.

Proof. First find an expansion M^3 of M^1 by suitable R, so $|\tau(M^3)| \leqq \mu$. Second find an expansion M^2 of M^3 which has Skolem functions and

$|\tau(M^2)| \leqq \mu$. Lastly we find H^2: just apply 3.8 for each $\eta \in \mathrm{Dom}\, H^1$ and apply 3.8 with $H^1(\eta)$, $(B^*_\eta, <)$,

$$\Theta_\eta = \{\varphi : \varphi \text{ an } L_{\omega,\omega}[\tau(M^1)]\text{-formula with parameters from } A^*_\eta\},$$

$$\Psi_\eta = \{\varphi : \varphi \text{ an } L_{\omega,\omega}[\tau(M^2)]\text{-formula with parameters from } A^*_\eta\}$$

here standing for $\langle a_i : < \alpha\rangle$, $(A, <)$, Θ, Ψ there. What we get ($\langle b_i : i < \alpha\rangle$) will be called $H^2(\eta)$.

Stage F — Claim. If (M^1, H^1) is an α-approximation $\beta + k \times m \leqq \alpha$, $n, m, k < \omega$ *then* there is a β-approximation (M^2, H^2) which is (n, m, k)-indiscernible and $(M^1, H^1) \leqq (M^2, H^2)$.

Proof. By the polarized partition relation of Erdos–Hajnal–Rado [EHR]. See representation (in our terminology) [Sh18] (or [Sh-a], AP) (or Erdos–Hajnal–Mate–Rado [EHRM]).

Stage G — Rest of the proof of Theorem 3.9. We use the style of [Sh18, §5] — see [Sh-a], VII, §5. Let χ be regular, large enough, such that $M^* \in H(\chi)$, $<^*_\chi$ a well ordering of $H(\chi)$, and let $A = \{i : i \leqq \mu\} \cup \tau(M^*)$.

Let $\mathfrak{A}^* = (H(\chi), \in, <^*_\chi, M^*, a)_{a \in A}$. There is a model $\mathfrak{A}$ of $\mathrm{Th}_{\mathscr{L}_{\omega,\omega}}(\mathfrak{A}^*)$ such that $\mathfrak{A}$ omits every type which $\mathfrak{A}^*$ omits and $\{t : \mathfrak{A}^* \vDash$ "t is an ordinal $<_{(2^\mu)^+}$"$\}$ is not well ordered. So there are $t_n \in \mathfrak{A}$ such that:

$$\vDash \text{“}t_n \text{ is an ordinal } <_{(2^\mu)^+}\text{”}$$

$$\vDash \text{“}t_{n+1} + (n+1)^2 < t_n\text{”}.$$

In $\mathfrak{A}$ we define by induction on n, Q_n, (M^n, H^n) such that:

(α) $\mathfrak{A} \vDash$ "(M^n, H^n) is a t_n-approximation",
(β) $\mathfrak{A} \vDash$ "$(M^n, H^n) \leqq (M^{n+1}, H^{n+1})$",
(γ) if $n = 2(m^2 + k)$, $k < m$ then (M^{n+1}, H^{n+1}) is (m, k)-indiscernible and (M^{n+1}, H^{n+1}) relates to (M^n, H^n) as in Stage E,
(δ) $R_n \in \tau(M^{n+1}) - \tau(M^n)$, for $n = 2m + 1$,
$R_n = R_{n+1} = \{(\eta, a) : a \in H^n(\eta)\}$.

For $n = 0$ use stage D.
For $n = l + 1$, l odd, use stage E to define (M^n, H^n).
For $n = l + 1$, l even, use stage F to define.

Now let $\tau_n = \tau(M_n)$, N_n be the Skolem Hull of $\varnothing$ in M^n (more exactly, M^n as interpreted in $\mathfrak{A}$). So $\tau(N_n) = \tau_n$, $\tau_n \subseteq \tau_{n+1}$, $|\tau_n| \leqq \mu$, $N_n \prec N_{n+1}$ (i.e. $N_n \prec (N_{n+1} \restriction \tau_n)$) and let $N_\omega = \bigcup_n N_n$, $\tau_\omega = \bigcup \tau_n$. So $N_n \prec N_\omega$.

We now define by induction on $n \leqq \omega$ a model N_n^* and Q_n such that:

(A) $N_0^* = N_\omega$, $\tau(N_n^*) = \tau_\omega$, $\| N_n^* \| = \mu$,

(B) $N_n^* \prec N_{n+1}^*$, $N_\omega^* = \bigcup_{n<\omega} N_n^*$,

(C) for $l \leqq n$, $P_0^{N_n^*} = P_l^{N_{n+1}^*}$,

(D) N_n^* is the Skolem Hull of

$$|N_\omega| \cup \left[\bigcup_{l \leqq n} P^{N_n^*} \right],$$

(E) $Q_n \subseteq P_n^{N_n^*}$,

(E) $Q_0 = P_0^{N_0^*}$,

(G) For every $\eta \in P_n^{N_n^*}$ (where N_η is defined as in (2) of the conclusion of 3.9):

$$\text{if } \eta \in Q_n \text{ then cf}\,[(N_{n+1}^*)_\eta, <^{N_{n+1}^*}] = \kappa_1,$$

$$\text{if } \eta \notin Q_n \text{ then cf}\,[(N_n^*)_\eta, <^{N_{n+1}^*}] = \kappa_2,$$

(H) for each $\eta \in P_n^{N_n^*} = P_n^{N_{n+1}^*}$,

$$((N_n^*)_\eta, Q) \in K_1 \cup K_2,$$

(I) for each $n < \omega$, every $L_{\omega,\omega}[\tau_n]$-type realized in N_n^* is realized in $N_0^* = N_\omega$, i.e. in some N_m, $m < \omega$.

The construction is straightforward: in the induction step use the indiscernibility built in [we can add $\| N_n^* \| = \mu$].

3.9a. Remark. We can get from the proof some additional information which does not seem useful. We can use $\delta(\mu)$ instead of $\beth_{(2^{|\mu|})^+}$, omit every type which M^* omits. We can, when constructing in N_{n+1}^*, assign arbitrary cofinalities to $(|N_{n+1}^*|, <^{N_{n+1}^*})$.

3.10. Theorem. *Supose in (our universe of set theory) V, for every α there is a measurable of order α (see e.g. Gitik* [G]). Then *in some generic extension V'*:

($*$) *for every regular $\mu < \lambda$: if $\mathscr{L}$ is a logic, $L(\exists^{\geqq\lambda}) \leqq \Delta(\mathscr{L})$ then* Beth$[\mathscr{L}, \Delta(L_{\infty,\mu})]$ *fails*.

Proof. Immediate by Gitik [G], but to clarify we deal with it lengthily. We use the following theorem of Gitik [G]:

($\oplus$) *if κ is a measurable of order θ, θ a regular cardinal $< \kappa$ and $\lambda < \kappa$*, then *for some forcing notions $P_{\kappa,\lambda,\theta}$, $\underset{\sim}{Q}_{\kappa,\lambda,\theta} (\in V^{P_{\kappa,\lambda,\theta}})$ the following holds*:

$P_{\kappa,\lambda,\theta}$ is λ-pseudo complete (see [Sh-b, Ch. X, §3]), *does not collapse cardinals, retains the regularity of κ.*

$Q_{\kappa,\lambda,\theta}$ is λ-pseudo complete, does not collapse cardinals, changes the cofinality of κ to θ and adds no new sequences of ordinals of length $< \theta$.

We also have (really follows generally, see [Sh250] 2.3, p. 6) minimal elements $\emptyset$ and order $\leqq_0$ witnessing to λ-pseudo completeness.

Suppose W is a set of triples $(\kappa, \lambda, \theta)$, of regular cardinals $\theta < \lambda < \kappa$ such that

$$\bigwedge_{l=1,2} (\kappa_l, \lambda_l, \theta_l) \in W \wedge \kappa_1 < \kappa_2 \Rightarrow \kappa_1 < \lambda_2.$$

Let $W[\kappa'] = \{(\kappa, \lambda, \theta) \in W : \kappa < \kappa'\}$. We define P_W as

$$\{\bar{p} : \bar{p} = \langle p_{(\kappa,\lambda,\theta)} : (\kappa, \lambda, \theta) \in W \rangle,$$

$$p_{(\kappa,\lambda,\theta)} \text{ is a } P_{W[\kappa]}\text{-name of a member of } P_{(\kappa,\lambda,\theta)} \text{ such that}$$

$$\{(\kappa, \lambda, \theta) : \not\Vdash 0 \leqq p_{(\kappa,\lambda,\theta)}\} \text{ is finite}\}$$

with obvious order (both $\leqq$ and $\leqq_0$).

Next $Q_W \in V^{P_W}$ is defined as:

$$\{\bar{q} : \bar{q} = \langle q_{(\kappa,\lambda,\theta)} : (\kappa, \lambda, \theta) \in W \rangle,$$

$$q_{(\kappa,\lambda,\theta)} \in (Q_{(\kappa,\lambda,\theta)})^{V^{P_{W[\kappa]}}} \text{ and}$$

$$\{(\kappa, \lambda, \theta) \in W : \neg\, 0 \leqq_0 q_{(\kappa,\lambda,\theta)}\} \text{ is finite}\};$$

order the obvious one ($\leqq$ and $\leqq_0$).

Now P_W and Q_W have the expected properties. We force with the iteration $\bar{R} = \langle R_i : i < \infty \rangle$ (support as in P_W) where for each i for some μ_i, $\bar{R} \restriction i \in H(\mu_i)$, $R_i \in \{P_{W_i}, P_{W_i^*}, Q_{W_i}\}$,

$$2^{\mu_i} < \operatorname{Min}\{\lambda : (\kappa, \lambda, \Theta) \in W_i\}.$$

The rest is left to the reader.

REFERENCES

[DSh65] K. Devlin and S. Shelah, *A weak form of the diamond which follows from* $2^{\aleph_0} < 2^{\aleph_1}$, Isr. J. Math. **29** (1978), 239–247.

[EHR] P. Erdos, A. Hajnal and R. Rado, *Partition relations for cardinal numbers*, Acta Math. Sci. Hung. **16** (1965), 193–196.

[EHMR] P. Erdos, A. Hajnal, A. Mate and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, North-Holland, Amsterdam, 1984.

[G] M. Gitik, *Changing cofinalities and the non-stationary ideal*, Isr. J. Math. **56** (1986), 280–314.

[HoSh109] W. Hodges and S. Shelah, *Infinite reduced products*, Ann. Math. Logic **20** (1981), 77–108.

[HoSh271] W. Hodges and S. Shelah, *There are reasonable nice logics*, J. Symb. Logic, to appear.

[Ma] J. A. Makowsky, *Compactness embeddings and definability*, in *Model Theoretic Logics* (J. Barwise and S. Feferman, eds.), Springer-Verlag, Berlin, 1985, pp. 645–716.

[MaSh62] J. A. Makowsky and S. Shelah, *The theorems of Beth and Craig in abstract logic I*, Trans. Am. Math. Soc. **256** (1979), 215–239.

[MaSh101] J. A. Makowsky and S. Shelah, *The theorems of Beth and Craig in abstract model theory, II, Compact Logics*, Proc. of a Workshop, Berlin, July 1977, Arch. Math. Logik **21** (1981), 13–36.

[MaSh116] J. A. Makowsky and S. Shelah, *Positive results in abstract model theory; a theory of compact logics*, Ann. Pure Appl. Logic **25** (1983), 263–299.

[MShS47] J. A. Makowsky, S. Shelah and J. Stavi, *Δ-logics and generalized quantifiers*, Ann. Math. Logic **10** (1976), 155–192.

[MkSh166] A. H. Mekler and S. Shelah, *Stationary logic and its friends, I*, Proc. of the 1980/1 Jerusalem Model Theory year, Notre Dame J. Formal Logic **26** (1985), 129–138.

[MkSh187] A. H. Mekler and S. Shelah, *Stationary logic and its friends II*, Notre Dame J. Formal Logic **27** (1986), 39–50.

[PiSh130] A. Pillay and S. Shelah, *Classification over a predicate I*, Notre Dame J. Formal Logic **26** (1985), 361–376.

[RuSh84] M. Rubin and S. Shelah, *On the elementary equivalence of automorphism groups of Boolean algebras, downward Skolem–Lowenheim theorems and compactness of related quantifiers*, J. Symb. Logic **45** (1980), 265–283.

[Sh-a] S. Shelah, *Classification Theory and the Number of Non-isomorphic Models*, North-Holland, Amsterdam, 1978, 542 pp. + xvi.

[Sh-b] S. Shelah, *Proper Forcing*, Springer Lecture Notes in Math. **940**, Springer-Verlag, Berlin, 1982, 496 pp. + xxiv.

[Sh8] S. Shelah, *Two cardinal compactness*, Isr. J. Math. **9** (1971), 193–198.

[Sh18] S. Shelah, *On models with power-like orderings*, J. Symb. Logic **37** (1972), 247–267.

[Sh43] S. Shelah, *Generalized quantifiers and compact logic*, Trans. Am. Math. Soc. **204** (1975), 342–364.

[Sh56] S. Shelah, *Refuting Ehrenfeucht conjecture on rigid models*, Proc. Symp. in memory of A. Robinson, Yale, 1975, Isr. J. Math. **25** (1976), 273–286.

[Sh72] S. Shelah, *Models wit second order properties I, Boolean algebras with no undefinable automorphisms*, Ann. Math. Logic. **14** (1978), 57–72.

[Sh107] S. Shelah, *Models with second order properties IV, A general method and eliminating diamonds*, Ann. Math. Logic **25** (1983), 183–212.

[Sh125] S. Shelah, *Consistency of* $\mathrm{Ext}(G, \mathbf{Z}) = Q$, Isr. J. Math. **39** (1981), 74–82.

[Sh129] S. Shelah, *On the number of non-isomorphic models of cardinality λ, $L_{\infty,\chi}$-equivalent to a fix model*, Notre Dame J. Formal Logic **22** (1981), 5–10.

[Sh133] S. Shelah, *On the number of non-isomorphic models in λ, $L_{\infty,\chi}$-equivalent to a fix model when λ is weakly compact*, Notre Dame J. Formal Logic **23** (1982), 21–26.

[Sh188] S. Shelah, *A pair of non-isomorphic models of power λ for λ singular with $\lambda^{\aleph_0} = \lambda$*, Notre Dame J. Formal Logic **25** (1984), 97–104.

[Sh189] S. Shelah, *On the possible number of* no(M) = *the number of non-isomorphic models $L_{\infty,\lambda}$ equivalent to M of power λ, for singular λ*, Notre Dame J. Formal Logic **26** (1985), 36–50.

[ShSn179] S. Shelah and C. Steinhorn, *On the axiomatizability by finitely many schemes*, Notre Dame J. Formal Logic **27** (1986), 1–11.

[ShSn180] S. Shelah and C. Steinhorn, *The non-axiomatizability of $L(Q_2^M)$*, Notre Dame J. Formal Logic, in press.

[Sh199] S. Shelah, *Remarks in abstract model theory*, Ann. Pure Appl. Logic **29** (1985), 255–288.

[Sh211] S. Shelah, *Stationary logic II: Comparison with other logics*, Notre Dame J. Formal Logic, submitted 1985.

[Sh220] S. Shelah, *Existence of many $L_{\infty,\lambda}$-equivalent non-isomorphic models of T of power λ*, Proc. Model Theory Conference Trento June 1986 (G. Cherlin and A. Marcja, eds.), Ann. Pure Appl. Logic **34** (1987), 291–310.

[Sh228] S. Shelah, *On the* no(M) *for M of singular power*, Lecture Notes in Math. **1182**, Springer-Verlag, Berlin, 1986, pp. 120–134.

[Sh250] S. Shelah, *Some notes on iterated forcing with $2^{\aleph_0} > \aleph_2$*, Notre Dame J. Formal Logic **29** (1988), 1–17.

[Sh355] S. Shelah, *$\aleph_{\omega+1}$ has a Jonsson algebra*, preprint (1988).